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LETTER TO THE EDITOR

**Quantum mechanics of relativistic particles in multiply connected spaces and the Aharonov-Bohm effect**

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**Abstract.** We consider the motion of free relativistic particles in multiply connected spaces. We show that if one of the spatial dimensions has the topology of a circle then the  $D$ -dimensional spacetime is compactified to  $D-1$  dimensions and the particle mass increases by an amount which is proportional to a quantum phase factor and inversely proportional to the radius of the circle. This provides a new non-perturbative mechanism of mass generation. We also consider the relativistic Aharonov-Bohm effect and we show that the propagator is similar in form to the non-relativistic one. The propagators are calculated without any approximation in both situations.

Quantum mechanics in multiply connected spaces is an old subject in theoretical physics [1-4] and it is essential for a proper understanding of the Aharonov-Bohm effect [5], fractional statistics [6], topological field theories [7] and probably high- $T_c$  superconductivity [8].

In the path integral framework the multiconnected character of the manifold does appear when we consider all paths which belong to distinct homotopic classes. Then the propagation amplitude to go from the point  $X_1$  to the point  $X_2$  is given by

$$G[X_2, X_1] = \sum_n \Xi_n G_n[X_2, X_1] \quad (1)$$

where  $\Xi_n$  is some unitary representation of the  $n$ th class of homotopy of the covering group of the manifold.

In the case of non-relativistic particles this problem has been studied by several authors [2, 3, 9-11] who have clarified the relation between the topological term  $\alpha\theta$  which we can add to the Lagrangian and the multivalued character of the wavefunction.

These aspects, together with a rigorous mathematical treatment of quantum mechanics in multiply connected spaces [2], were essential for recent developments in fractional statistics and topological quantum field theories.

However, despite the large amount of work in this area, there is no discussion, to our knowledge, of the motion of relativistic particles in multiply connected spaces. Probably this gap is due to the fact that only recently a complete understanding of the quantization of the relativistic particle in terms of path integrals was reached. The crucial point is that the reparametrization invariance makes the path integral non-trivial

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and the Faddeev-Popov [12] or Batalin-Fradkin-Vilkovisky [13] technique must necessarily be used.

Technically the problem of quantization of a generally covariant theory has to deal with the trouble of how to fix the gauge. The gauge fixing, as was discussed some time ago by Teitelboim [14], must be consistent with the complete fixation of the gauge parameters at the initial and final points of the trajectory. Therefore, in this class of theories, we are led naturally to non-canonical gauge choices, like the proper time gauge.

The purpose of this paper is to discuss two examples involving free relativistic particles moving on spaces with non-trivial topology using the path integral method. In particular, the analogous of the problem solved by Schulman [1, 3] for a free non-relativistic particle moving on a space with the topology of a circle  $S^1$  will be solved explicitly in the proper time gauge. We will show that when we impose this topology the propagator of the relativistic particle is compactified from  $D$  to  $D-1$  dimensions generating a mass proportional to the inverse radius of the circle and proportional to a quantum phase. This provides a new mechanism of dynamical mass generation by non-perturbative methods.

We then consider the relativistic Aharonov-Bohm effect. We will present a procedure which allow us to obtain an explicit expression for the propagator in momentum space. Finally we compare our results with the non-relativistic ones showing that the interference pattern is the same in both cases.

Let us consider a relativistic particle with mass  $m$  in a  $D$ -dimensional spacetime described by the following action

$$S = \int_{t_1}^{t_2} d\tau (P^\mu \dot{X}_\mu - N \mathcal{H}) \quad (2)$$

where  $N$  is the Lagrange multiplier and  $\mathcal{H}$  is the first class constraint defined by

$$\mathcal{H} = \frac{1}{2}(P^2 + m^2) \quad (3)$$

and  $P_\mu$  is the momentum conjugated to  $X_\mu$ .

The reparametrization invariance is given by the following transformations

$$\begin{aligned} \delta X^\mu &= \varepsilon P^\mu \\ \delta P^\mu &= 0 \\ \delta N &= \dot{\varepsilon} \end{aligned} \quad (4)$$

which leave invariant the action (2) if at the endpoints we have

$$\varepsilon(t_1) = \varepsilon(t_2) = 0. \quad (5)$$

As has been discussed in [14] a gauge choice compatible with (4) is the proper time gauge

$$\dot{N} = 0. \quad (6)$$

To quantize the theory we will use the path integral formalism. The propagation amplitude for the particle to go from  $X_1$  to  $X_2$  is

$$G[X_2, X_1] = \int \mathcal{D}N \mathcal{D}X_\mu \mathcal{D}P^\mu \delta[\dot{N}] \det(\partial_\tau^2) \exp\left(i \int_{t_1}^{t_2} d\tau (P^\mu \dot{X}_\mu - N \mathcal{H})\right). \quad (7)$$

The formal expression (7) must be evaluated specifying the boundary conditions, i.e. in this case the topology of  $S^1$ .

Let us assume that the topology of  $S^1$  is imposed only in one of the coordinates, say  $X^1$ . Then we have the boundary conditions

$$\begin{aligned} X^0(t_1) &= X_1^0 & X^0(t_2) &= X_2^0 \\ X^1(t_1) &= X_1^1 & X^1(t_2) &= X_2^1 + 2\pi n l \\ X^i(t_1) &= X_1^i & X^i(t_2) &= X_2^i \end{aligned} \tag{8}$$

where  $i = 2, \dots, D-1$ , and  $l$  is the radius of the circle and  $n$  an integer.

The path integral (7) can now be evaluated explicitly. Notice that in (7) the factor  $\delta[\dot{N}]$  states that only the zero mode ( $N(0)$ ) of  $N(t)$  contributes to the path integral and therefore the functional measure  $\mathcal{D}N$  can be replaced by an ordinary measure  $dN(0)$ . The integration limits for  $N(0)$ , by causality requirements [14], is from 0 to  $\infty$ . The Euclideanized determinant  $\det(-\partial_\tau^2)$  can be evaluated by using the  $\zeta$ -function regularization after we impose that the eigenfunctions of  $-\partial_\tau^2$  vanish at the endpoints of the trajectory†, the result being  $\det(-\partial_\tau^2) = \Delta t = t_2 - t_1$ .

Then (7) becomes

$$G_n[X_2, X_1] = \int_0^\infty dT \int \mathcal{D}X_\mu \mathcal{D}P^\mu \exp\left(i \int_{t_1}^{t_2} d\tau (P^\mu \dot{X}_\mu - N(0)\mathcal{H})\right) \tag{9}$$

where  $T = N(0)\Delta t$ . In (9) we added a lower index  $n$  to  $G$  which means that  $G_n[X_2, X_1]$  is the propagation amplitude for the  $n$ th class of homotopy.

If we now write  $P^\mu \dot{X}_\mu$  as  $(d/dt)(P^\mu X_\mu) - \dot{P}^\mu X_\mu$  and we integrate on  $X_\mu$  we get a factor of  $\delta[P_\mu]$  which states that the momentum is conserved and allows the functional integral in  $P_\mu$  to become an ordinary integral.

Using the boundary conditions (8) in order to evaluate the surface term, we get that (9) becomes

$$\begin{aligned} G_n[X_2, X_1] &= \int_0^\infty dT \int \frac{d^D P}{(2\pi)^D} \exp(iP^\mu \Delta X_\mu + 2in\pi l P_1 - iT\mathcal{H}) \\ &= \int \frac{d^D P}{(2\pi)^D} \frac{\exp(iP^\mu \Delta X_\mu + 2inl P_1)}{P^2 + m^2 - i\epsilon} \end{aligned} \tag{10}$$

where the factor  $i\epsilon$  was introduced to guarantee the convergence of (9).

Using (1) we obtain the complete propagator as

$$G[X_2, X_1] = \sum_n \Xi_n \int \frac{d^D P}{(2\pi)^D} \frac{\exp(iP^\mu \Delta X_\mu + 2inl P_1)}{P^2 + m^2 - i\epsilon}. \tag{11}$$

Following the arguments presented by Schulman [1] the  $\Xi_n$  can be easily obtained by observing that (11) depends only on the difference  $X_2 - X_1$ , and so it is easy to show that  $\Xi_n = e^{i\delta n}$ , where  $\delta$  is a phase whose origin is strictly quantic and induced

† This has a natural explanation in the BfV formalism. In fact to impose that the eigenfunctions of  $-\partial_\tau^2$  vanish at the endpoints of the trajectory is equivalent to the imposition of homogeneous boundary conditions for the ghosts.

by the non-trivial topology of the coordinate  $X^{1\dagger}$ . Consequently (11) is

$$\begin{aligned}
 G[X_2, X_1] &= \sum_{n=-\infty}^{\infty} \int \frac{d^D P}{(2\pi)^D} \frac{\exp(iP^\mu \Delta X_\mu + in(2\pi l P_1 + \delta))}{P^2 + m^2 - i\epsilon} \\
 &= (2\pi) \int \frac{d^D P}{(2\pi)^D} \frac{e^{iP^\mu \Delta X_\mu}}{P^2 + m^2 - i\epsilon} \delta[2\pi l P_1 + \delta] \\
 &= \exp\left(\frac{-i\delta \Delta X_1}{2\pi l}\right) \int \frac{d^{D-1} P}{(2\pi)^{D-1}} \frac{e^{iP^\mu \Delta X_\mu}}{P^2 + \tilde{m}^2 - i\epsilon} \quad (12)
 \end{aligned}$$

where now  $\mu = 0, 2, \dots, D-1$ . Equation (12) is the propagator for a particle of mass  $\tilde{m}^2 = m^2 + \delta^2/(2\pi l)^2$  moving in a  $(D-1)$ -dimensional spacetime with trivial topology. Therefore, by imposing a non-trivial topology in one of the spatial coordinates of the  $D$ -dimensional spacetime we generate a compactification to  $D-1$  dimensions. This has no analogy with the non-relativistic case. It is also not analogous to the Cremmer-Scherk [15] compactification since our results are derived from a quantum phase  $\delta$  and in this sense it is not equivalent to the compactification mechanism of the Kaluza-Klein theories [16] or string theories [17]. It also gives a new non-perturbative mechanism of mass generation. It would be very interesting to extend these ideas to quantum field theories‡

We now consider the Aharonov-Bohm effect. This is a two-dimensional situation where the electron moves in a multiply connected space due to the presence of a cylinder of infinite length filled with a magnetic field. The relativistic effects should be observable by increasing the velocity of the electron. However, it seems that there is no experimental investigation in this direction [20].

Topologically speaking the situation described in figure 1 is not the most general possible since the electrons can follow other paths, e.g., going around the cylinder (see figure 2), and the problem is how to incorporate this into the path integral.

In principle it would be possible to extend the method developed by Inomota-Singh [21] and Gerry-Singh [22] to solve the non-relativistic Aharonov-Bohm effect using the path integral. However, it seems more convenient to write directly the path integral for the relativistic problem and to factorize the corresponding integral in its space and time components (regarding the position vector  $X^\mu$ ).

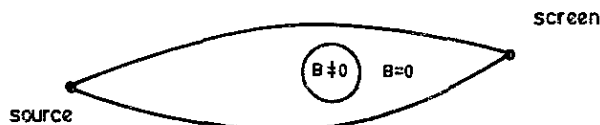


Figure 1.

† Here  $\delta$  is a quantum phase that depends on how the singularity is constructed, e.g. if this singularity is constructed with an infinite solenoid filled with a magnetic field (as in the Aharonov-Bohm effect) this phase can be interpreted as the magnetic flux

‡ The literature about quantum field theories in multiply connected spaces is very restricted. Perhaps the first paper on this subject is due to Dowker [4]. More recently non-Abelian gauge theories on  $S^1$  and  $S^2$  were shown to have a dynamical mass generation at the one loop level [18], the same as that occurring in the Schwinger model [19]

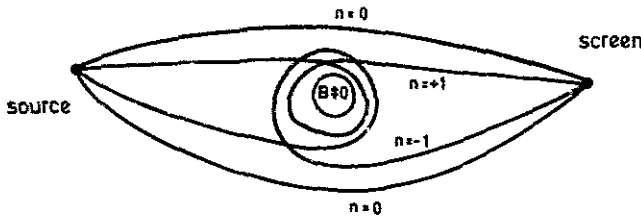


Figure 2.

The integral in  $X^0$  can be solved easily and the spatial integrals can be evaluated by using polar coordinates (see figure 3) as is done in [21] and [22]. Then, the formal path integral for the  $n$ th class of homotopy in the proper time gauge has the following form

$$G_n[X_2, X_1] = \int \mathcal{D}N \mathcal{D}X^\mu \det(N)^{-1} \delta[N] \det(N\partial_\tau^2) \times \exp\left(i \int_{t_1}^{t_2} d\tau \left(\frac{1}{2N} \dot{X}^2 + \frac{m^2}{2} N\right)\right). \tag{13}$$

The amplitude (13) is a Lagrangian expression. The factor  $\det(N^{-1})$  has been incorporated to make the functional measure invariant under reparametrizations. It is also easy to see, by using the  $\zeta$ -function regularization, that  $\det(N\partial_\tau^2)$  is factorizable and therefore the factors  $\det(N^{-1})$  and  $\det(N)$  cancel among themselves. As before we have  $\det(-\partial_\tau^2) = \Delta t$ .

The factor  $\delta[N]$  allows us to replace the path integral in  $N$  by an ordinary integral in  $N(0)$  with integration limits from 0 to  $\infty$ . Then (13) becomes

$$G_n[X_2, X_1] = \int_0^\infty dT \exp\left(i \frac{m^2 T}{2}\right) \int \mathcal{D}X^0 \exp\left(i \int_{t_1}^{t_2} d\tau \frac{-\dot{X}_0^2}{2N(0)}\right) \times \int \mathcal{D}^2 X \exp\left(i \int_{t_1}^{t_2} d\tau \frac{\dot{X}^2}{2N(0)}\right). \tag{14}$$

To solve the integrals in  $X^0$  and  $X$  we consider that they are formally equal to the integrals for the non-relativistic particle in one and two dimensions with mass  $m = N(0)^{-1}$ . So the  $X^0$  integral is simply

$$\left(\frac{1}{2\pi T}\right)^{1/2} \exp\left(\frac{-i(\Delta X_0)^2}{T}\right). \tag{15}$$

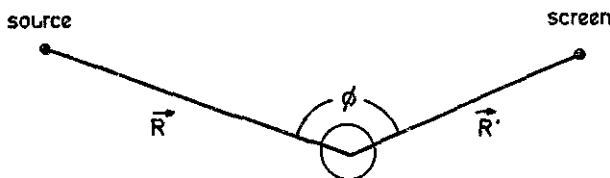


Figure 3.

The  $X$  integral is more complicated because we have to take into account that the particle can go around the cylinder. This can be done and we refer to the papers [21] and [22] for details (see also [23]). The result for the integral (14) is

$$G_n[X_2, X_1] = \int_0^\infty dT \int_{-\infty}^{+\infty} d\lambda \frac{1}{(2\pi T)^{3/2}} \times \exp\left(\frac{im^2 T}{2} - i \frac{(\Delta X_0)^2}{2T} + \frac{i}{2} \frac{(R^2 + R'^2)}{T} + i\lambda(\phi + 2\pi n)\right) I_{|\lambda|}\left(i \frac{RR'}{T}\right)$$

where  $\phi$  is the angle difference between  $R$  and  $R'$  (see figure 2) and  $I_{|\lambda|}$  is the modified Bessel function.

Following the same arguments used earlier to determine  $\Xi_n$  in (1) we obtain the complete propagator

$$G[X_2, X_1] = \int_0^\infty dT \frac{1}{(2\pi T)^{3/2}} \exp\left(\frac{im^2 T}{2} - i \frac{(\Delta X_0)^2}{2T} + \frac{i}{2} \frac{(R^2 + R'^2)}{T}\right) \times \sum_{n=-\infty}^\infty (-1)^{|n+\alpha|} \exp(-i(n+\alpha)\phi) J_{|n+\alpha|}\left(\frac{RR'}{T}\right) \tag{16}$$

where in this case the phase is  $\alpha = e\Phi$ , where  $\Phi$  is the magnetic flux and  $e$  is the electric charge. In (16) we made use of the identity

$$I_\nu(ix) = (i)^\nu J_\nu(x).$$

Equation (16) can also be written explicitly in momentum space

$$G[X_2, X_1] = \int d^3P \int_0^\infty dT \exp\left(iP^\mu \Delta X_\mu - i \frac{T}{2} (P^2 + m^2)\right) \times \sum_{n=-\infty}^\infty (-1)^{|n+\alpha|} \exp(-i(n+\alpha)\phi) J_{|n+\alpha|}\left(\frac{RR'}{T}\right) \tag{17}$$

Making the change of variables  $T = 1/z$ , using the identity

$$J_{\nu-1}(x) + J_{\nu+1}(x) = 2\nu J_\nu(x)$$

( $\nu$  arbitrary) and using the integral [24]

$$\int_0^\infty dz z^{-1} e^{\alpha/z} J_\nu(\beta z) = 2J_\nu(\sqrt{\alpha\beta}) K_\nu(\sqrt{\alpha\beta}) \quad \text{Re}(\alpha) > 0, \beta > 0$$

it follows that (16) has the following form

$$G[X_2, X_1] = \lim_{\varepsilon \rightarrow 0} \sum_{n=-\infty}^\infty \frac{(-1)^{|n+\alpha|}}{|n+\alpha|} e^{-i(n+\alpha)\phi} \int \frac{d^3P}{(2\pi)^3} e^{iP^\mu \Delta X_\mu} \times (J_{|n+\alpha|-1}(\sqrt{RR'\rho}) + J_{|n+\alpha|+1}(\sqrt{RR'\rho}))(K_{|n+\alpha|-1}(\sqrt{RR'\rho}) + K_{|n+\alpha|+1}(\sqrt{RR'\rho})) \tag{18}$$

with  $\rho = \varepsilon - i(P^2 + m^2)$ .

We have formulated the path integral quantization of a relativistic particle on a spacetime with non-trivial topology and written explicitly expressions for the propagators. For the free relativistic particle we have found a new dynamical non-perturbative mechanism of mass generation and it would be very interesting to find an analogous mechanism in quantum field theories.

We also discussed the relativistic Aharonov-Bohm effect and calculated the propagator. We remark that its form is the same as the non-relativistic one. If we write

$$G[X_2, X_1] = \sum_{n=-\infty}^{\infty} (-i)^{|n+\alpha|} \exp(-i(n+\alpha)\phi) F_{|n+\alpha|} \quad (19)$$

we recover the relativistic propagator (17) for

$$F_{|n+\alpha|} = \int d^3P \int_0^{\infty} dT \exp\left(iP^\mu \Delta X_\mu - i\frac{T}{2}(P^2 + m^2)\right) J_{|n+\alpha|}\left(\frac{RR'}{T}\right) \quad (20)$$

while for the non-relativistic case [23],

$$F_{|n+\alpha|} = \frac{m}{2\pi i} \exp\left(\frac{2mi}{\tau}(R^2 + R'^2)\right) J_{|n+\alpha|}\left(\frac{mRR'}{\tau}\right) \quad (21)$$

where  $\tau$  is the time that the particle takes to go from  $X_1$  to  $X_2$ .

This fact shows that we have a universal form for the propagator of the Aharonov-Bohm effect. If we had considered the relativistic spinning particle, for example, only the factor  $F_{|n+\alpha|}$  in (19) would change. If we now apply the Poisson summation formula to (19) and make a change of variables [25] we can rewrite (19) as

$$G[X_2, X_1] = \sum_{m=-\infty}^{\infty} T_m e^{-2\pi i m \alpha} \quad (22)$$

where

$$T_m = \int_{-\infty}^{+\infty} d\lambda (-i)^{|\lambda|} e^{-i\lambda\phi} F_{|\lambda|} \quad (23)$$

so that all dependence on the flux  $\Phi$  is in the exponential factor in (22). This means then that the interference pattern will be the same whichever particle we take, relativistic, spinning particle, etc. The interference pattern of the Aharonov-Bohm effect is universal since it depends only on the magnetic flux.

Of course, this requires a note of caution, in fact, recently Hagen [26] has suggested that the spin could modify the interference pattern if the initial beam of particles is polarized. (Another derivation of this result based on the use of the conservation of helicity is given in [27]) We also assumed that the cylinder is infinite in length but it is known that a finite cylinder changes the usual results [28].

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## References

- [1] Schulman L S 1968 *Phys Rev* **176** 1558
- [2] Laidlaw M G G and deWitt C M 1971 *Phys Rev D* **3** 1375
- [3] Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)
- [4] Dowker J S 1972 *J. Phys. A: Math. Gen* **5** 304
- [5] Aharonov Y and Bohm D 1959 *Phys. Rev* **115** 485
- [6] Wilczek F 1990 *Princeton preprint*
- [7] Witten E 1988 *Commun. Math. Phys.* **117** 353
- [8] Chen Y-H, Wilczek F, Witten E and Halperin B I 1989 *Int. J. Mod. Phys. B* **3** 1001
- [9] Rothe H J 1981 *Nuovo Cimento* **62A** 54
- [10] Rothe H J and Swieca J A 1979 *Nucl. Phys. B* **149** 237



- [11] Lee B H, Lee S H, Weinberg E J and Lee K 1988 *Phys. Rev. Lett.* **60** 2231
- [12] Faddeev L D and Slavnov A A 1980 *Introduction to Quantum Theory: Gauge Fields* (Reading, MA Benjamin)
- [13] Henneaux M 1985 *Phys. Rep.* **126** 1
- [14] Teitelboim C 1982 *Phys. Rev. D* **25** 3152
- [15] Cremmer E and Scherk J 1977 *Nucl. Phys. B* **110** 61
- [16] Duff M, Nilsson B E W and Pope C N 1986 *Phys. Rep.* **130** 1
- [17] Green M B, Schwarz J H and Witten E 1987 *Superstring Theory* (Cambridge Cambridge University Press)
- [18] Hosotani Y 1983 *Phys. Lett.* **126B** 309, **129B** 193, 1989 *Ann. Phys., NY* **190** 233  
Toms D 1983 *Phys. Lett.* **126B** 445  
Hetrick J E and Hosotani Y 1989 *Phys. Lett.* **230B** 88  
Rajeev S G 1989 *Preprint*. UTP-1069
- [19] Manton N S 1985 *Ann. Phys., NY* **159** 220
- [20] Olariu S and Popescu I 1985 *Rev. Mod. Phys.* **57** 339
- [21] Inomata A and Singh V A 1978 *J. Math. Phys.* **19** 2318
- [22] Gerry C and Singh V A 1979 *Phys. Rev. D* **20** 2550
- [23] Shiekh A Y 1986 *Ann. Phys., NY* **166** 299
- [24] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series and Products* (New York Academic)
- [25] Berry M V 1980 *Eur. J. Phys.* **1** 1240
- [26] Hagen C R 1990 *Preprint* UR-1145
- [27] Vera F and Schmidt I 1990 *Phys. Rev. D* **42** 3591
- [28] Afanesiev G N and Shilov V M 1989 *J. Phys. A: Math. Gen.* **22** 5195